

LARGE FOURIER QUASICRYSTALS AND WIENER'S THEOREM

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Abstract. We find new simple conditions for support of a discrete measure on Euclidean space to be a finite union of translated lattices. The arguments are based on a local analog of Wiener's Theorem on absolutely convergent trigonometric series and theory of almost periodic functions.

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1. INTRODUCTION

Denote by $S(\mathbb{R}^d)$ the Schwartz space of test functions $\varphi \in C^\infty(\mathbb{R}^d)$ with finite norms

$$p_m(\varphi) = \sup_{\mathbb{R}^d} (\max\{1, |x|\})^m \max_{|k_1|+\dots+|k_d|\leq m} |D^k(\varphi(x))|, \quad m = 0, 1, 2, \dots,$$

$k = (k_1, \dots, k_d) \in (\mathbb{N} \cup \{0\})^d$, $D^k = \partial_{x_1}^{k_1} \dots \partial_{x_d}^{k_d}$. These norms generate a topology on $S(\mathbb{R}^d)$, and elements of the space $S'(\mathbb{R}^d)$ of continuous linear functionals on $S(\mathbb{R}^d)$ are called tempered distributions. For every tempered distribution f there exist $C > 0$ and $m \in \mathbb{N} \cup \{0\}$ such that for all $\varphi \in S(\mathbb{R}^d)$

$$|f(\varphi)| \leq Cp_m(\varphi).$$

Moreover, this estimate is sufficient for distribution f to be in $S'(\mathbb{R}^d)$ (see [16], Ch.3).

The Fourier transform of a tempered distribution f is defined by the equality

$$\hat{f}(\varphi) = f(\hat{\varphi}) \quad \text{for all } \varphi \in S(\mathbb{R}^d),$$

where

$$\hat{\varphi}(y) = \int_{\mathbb{R}^d} \varphi(x) \exp\{-2\pi i \langle x, y \rangle\} dm_d(x)$$

is the Fourier transform of the function φ . Note that the Fourier transform of every tempered distribution is also a tempered distribution. Here we consider only the case when f and \hat{f} are measures on \mathbb{R}^d .

To formulate the results of the paper we need some notions and definitions.

Set $B(x, r) = \{y \in \mathbb{R}^d : |y - x| < r\}$, $B(r) = B(0, r)$. We say that $E \subset \mathbb{R}^d$ is a *full-rank lattice* if $E = T\mathbb{Z}^d$ for some nondegenerate linear operator T on \mathbb{R}^d ; E is *relatively dense* if there exists $R < \infty$ such that $B(x, R) \cap E \neq \emptyset$ for each $x \in \mathbb{R}^d$; E is *discrete* if $E \cap B(x, 1)$ is finite for all $x \in \mathbb{R}^d$; E is *uniformly discrete* if $|x - x'| \geq \varepsilon > 0$ for all $x, x' \in E, x \neq x'$.

A Radon measure μ is *discrete* (uniformly discrete) if its support is discrete (uniformly discrete), μ is *translation bounded* if its variations $|\mu|$ are uniformly bounded on balls of radius 1, and μ is *slowly increasing* if $|\mu|(B(r))$ grows at most polynomially as $r \rightarrow \infty$. Note that every translation bounded measure is slowly increasing, and every slowly increasing measure belongs to $S'(\mathbb{R}^d)$.

Following [9] we say that μ is a *Fourier quasicrystal* if both measures $\mu, \hat{\mu}$ are slowly increasing atomic measures. More precisely, we will suppose that

$$\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda, \quad \hat{\mu} = \sum_{\gamma \in \Gamma} b_\gamma \delta_\gamma, \quad \log \left[\sum_{|\lambda| < r} |a_\lambda| + \sum_{|\gamma| < r} |b_\gamma| \right] = O(\log r), \quad r \rightarrow \infty,$$

where δ_z means the unit mass at the point z and the sets Λ, Γ are countable. In the paper we usually consider the case of discrete *support* Λ and countable *spectrum* Γ of μ . We say that μ is a *large Fourier quasicrystal* if, in addition, $\inf_{\lambda \in \Lambda} |a_\lambda| > 0$. The simplest example of a Fourier quasicrystal is the measure $\mu_0 = \sum_{k \in \mathbb{Z}^d} \delta_k$. By the Poisson formula we have $\hat{\mu}_0 = \mu_0$.

Such measures are the main object in the theory of Fourier quasicrystals (see [2]-[12]). The corresponding notions were inspired by experimental discovery of non-periodic atomic structures with diffraction patterns consisting of spots, which was made in the mid '80s.

In the paper we are interested in the cases when support Λ is a subset of a finite union of translated full-rank lattices. Several very interesting results of this type were obtained by N.Lev, A.Olevskii in [8], [9]. Here it is one of them.

Theorem 1 ([9]). *Let μ be a Fourier quasicrystal on \mathbb{R}^d with discrete spectrum Γ and uniformly discrete set of differences $\Lambda - \Lambda$. Then Λ is a subset of a finite union of translates of a single full-rank lattice L , and Γ is a subset of a finite union of translates of the conjugate lattice.*

Also, there exists a Fourier quasicrystal with countable spectrum Γ such that $\Lambda - \Lambda$ is uniformly discrete, but Λ is not contained in a finite union of translates of any lattice.

We prove the following theorem, which amplifies the previous one

Theorem 2. *Let μ be a large Fourier quasicrystal on \mathbb{R}^d with a discrete set of differences $\Lambda - \Lambda$. Then Λ is a finite union of translates of a single full-rank lattice L .*

In the next section we will give a generalization of Theorem 2 for pairs of Fourier quasicrystals.

Results of another type of results were obtained by Y.Meyer and A.Cordoba.

Theorem 3 (Y.Meyer [11]). *Let $\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda$ be a measure with discrete support Λ and $a_\lambda \in S$ for some finite set $S \subset \mathbb{C} \setminus \{0\}$. If $\mu \in S'(\mathbb{R})$ and if its Fourier transform $\hat{\mu}$ is a translation bounded measure on \mathbb{R} , then $\Lambda = E \Delta \bigcup_{j=1}^N (\alpha_j \mathbb{Z} + \beta_j)$, where $\alpha_j > 0$, $\beta_j \in \mathbb{R}$, and the set E is finite.*

Here $A \Delta B$ means the symmetrical difference between A and B .

In [5] M.Kolountzakis extended the above theorem to measures on \mathbb{R}^d . He replaced the condition "the measure $\hat{\mu}$ is translation bounded" with the weaker one

$$(1) \quad |\hat{\mu}|(B(r)) = O(r^d) \quad \text{as } r \rightarrow \infty.$$

He also found a condition for support μ to be a finite union of several full-rank lattices. His result is very close to Cordoba's one:

Theorem 4 ([2]). *Let μ be a uniformly discrete Fourier quasicrystal on \mathbb{R}^d with a_λ belonging to a finite set F . If the measure $\hat{\mu}$ is translation bounded, then Λ is a finite union of translates of several, possibly incommensurable, full-rank lattices.*

In paper [4] we replaced the conditions " a_λ from a finite set" by " $|a_\lambda|$ from a finite set" and the condition " $\hat{\mu}$ is translation bounded" by (1).

Now we obtain the following theorem.

Theorem 5. *Let μ be a uniformly discrete large Fourier quasicrystal on \mathbb{R}^d , and let $\hat{\mu}$ satisfy (1). Then Λ is a finite union of translates of several disjoint full-rank lattices.*

The proof is based on an analog of Wiener's Theorem on Fourier series:

Theorem 6 (N.Wiener, see, for example, [17], Ch.VI). *Let $F(t) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n t}$ be an absolutely convergent Fourier series, and $h(z)$ be a holomorphic function on a neighborhood of the set $\overline{\{F(t) : t \in [0, 1]\}}$. Then the function $h(F(t))$ admits an absolutely convergent Fourier series expansion as well.*

Denote by W the class of absolutely convergent series $\sum_n c_n e^{2\pi i \langle x, \gamma_n \rangle}$, $\gamma_n \in \mathbb{R}^d$. We prove the following theorem

Theorem 7. *Let $K \subset \mathbb{C}$ be an arbitrary compact, $h(z)$ be a holomorphic function on a neighborhood of K , and $f \in W$. Then there is a function $g \in W$ such that if $f(x) \in K$ then $g(x) = h(f(x))$.*

If $K = \overline{f(\mathbb{R}^d)}$ then we obtain the global Wiener's Theorem for almost periodic functions.

Finally, we prove an analog of Wiener's Theorem for Fourier quasicrystals:

Theorem 8. *Let $\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda$ be a uniformly discrete Fourier quasicrystal on \mathbb{R}^d with the translation bounded measure $\hat{\mu}$. Suppose that $h(z)$ is a holomorphic function on a neighborhood of the closure of the set $\{a_\lambda : \lambda \in \Lambda\}$. Then $\nu = \sum_{\lambda \in \Lambda} a_\lambda h(a_\lambda) \delta_\lambda$ is also Fourier quasicrystal with translation bounded $\hat{\nu}$.*

Corollary. *Let $\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda$ be a uniformly discrete Fourier quasicrystal on \mathbb{R}^d with translation bounded measure $\hat{\mu}$. Let A be a connected component of the set $\overline{\{a_\lambda : \lambda \in \Lambda\}}$ such that $0 \notin A$. Then $\{\lambda \in \Lambda : a_\lambda \in A\}$ is a finite union of translates of several full-rank lattices.*

2. LARGE QUASICRYSTALS AND ALMOST PERIODIC MEASURES

We recall some definitions related to the notion of almost periodicity (see, for example, [10]).

A continuous function f on \mathbb{R}^d is *almost periodic*, if for every $\varepsilon > 0$ the set of ε -almost periods of f

$$\{\tau \in \mathbb{R}^d : \sup_{x \in \mathbb{R}^d} |f(x + \tau) - f(x)| < \varepsilon\}$$

is a relatively dense set in \mathbb{R}^d .

For every almost periodic function f there exists the limit uniform in $x \in \mathbb{R}^d$

$$\mathcal{M}(f) = \lim_{r \rightarrow \infty} \frac{1}{\omega_d r^d} \int_{B(x, r)} f(y) dm_d(y),$$

where ω_d is volume of the unit ball. The Fourier coefficient of f at frequency $\gamma \in \mathbb{R}^d$ is

$$a_f(\gamma) = \mathcal{M}(f(x) e^{-2\pi i \langle x, \gamma \rangle}).$$

For every f only countably many $a_f(\gamma)$ do not vanish, and the set $\{a_f(\gamma)\}$ defines the function f uniquely.

It can be easily proved that every function $f = \sum_n c_n e^{2\pi i \langle x, \gamma_n \rangle} \in W$ is almost periodic. We set $\|f\|_W = \sum_n |c_n|$.

$$\text{Since } \mathcal{M}(e^{2\pi i \langle x, \gamma \rangle} e^{2\pi i \langle x, -\gamma' \rangle}) = \begin{cases} 1, & \gamma = \gamma', \\ 0, & \gamma \neq \gamma', \end{cases} \quad \text{we get } a_f(\gamma) = \begin{cases} c_n, & \gamma = \gamma_n, \\ 0, & \gamma \neq \gamma_n. \end{cases}$$

Moreover, $\hat{f} = \sum_n c_n \delta_{\gamma_n}$. Indeed, if $\psi \in S(\mathbb{R}^d)$, then

$$(\hat{f}, \psi) = (f, \hat{\psi}) = \sum_n c_n \int_{\mathbb{R}^d} e^{2\pi i \langle y, \gamma_n \rangle} \hat{\psi}(y) dy = \sum_n c_n \psi(\gamma_n).$$

Next, a (complex) measure μ on \mathbb{R}^d is *almost periodic*, if for every continuous function ψ on \mathbb{R}^d with compact support the function $(\psi \star \mu)(x)$ is almost periodic in $x \in \mathbb{R}^d$. It is clear that support of every almost periodic measure is relatively dense in \mathbb{R}^d .

A discrete set Λ is almost periodic, if measure $\sum_{\lambda \in \Lambda} \delta_\lambda$ is almost periodic.

Theorem 9 (L.Ronkin [14]). *Every almost periodic measure is translation bounded.*

A connection between almost periodicity of measure and properties of its Fourier transform was obtained by Y.Meyer:

Theorem 10 ([10]). *Let μ and its Fourier transform $\hat{\mu}$ be translation bounded measures. Then μ is almost periodic iff spectrum of μ is countable.*

Here we need an extension of this result.

Theorem 11. *Let μ be a measure from $S'(\mathbb{R}^d)$, and let $\hat{\mu}$ be a slowly increasing measure. Then μ is almost periodic iff it is translation bounded and has a countable spectrum.*

First we prove some lemmas.

Lemma 1. *Let μ be a measure from $S'(\mathbb{R}^d)$ with countable spectrum, let $\hat{\mu}$ be a slowly increasing measure, and let $\psi \in S(\mathbb{R}^d)$. Then convolution $\psi \star \mu$ belongs to W .*

Proof. Let $\hat{\mu} = \sum_{\gamma \in \Gamma} b_\gamma \delta_\gamma$. We have

$$(2) \quad (\psi \star \mu)(s) = \int \psi(s - x) d\mu(x) = \int \hat{\psi}(y) e^{2\pi i \langle y, s \rangle} d\hat{\mu}(y) = \sum_{\gamma \in \Gamma} b(\gamma) \hat{\psi}(\gamma) e^{2\pi i \langle s, \gamma \rangle}.$$

Since $\hat{\psi} \in S(\mathbb{R}^d)$, we get

$$\int |\hat{\psi}(y)| d|\hat{\mu}|(y) \leq \int_0^\infty C \min\{1, t^{-N}\} dM(t) = C \lim_{t \rightarrow \infty} \frac{M(t)}{t^N} + CN \int_1^\infty \frac{M(t)}{t^{N+1}} dt,$$

where $M(t) = |\hat{\mu}|(B(t))$. For appropriate N the latter integral is finite. Therefore the sum in (2) converges absolutely. ■

Lemma 2. *Let μ be a translation bounded measure, and let $\hat{\mu}$ be a slowly increasing measure. Then for each $\lambda \in \mathbb{R}^d$ the limit*

$$\lim_{R \rightarrow \infty} \frac{1}{\omega_d R^d} \int_{B(R)} e^{-2\pi i \langle x, \lambda \rangle} d\mu(x)$$

exists and equals $\hat{\mu}(\{\lambda\})$.

Proof. Pick an arbitrary $\varepsilon > 0$. Let $\psi(|x|)$ be C^∞ -differentiable function on \mathbb{R}^d such that $\psi(t) = 1/\omega_d$ for $t < 1 - \varepsilon$, $\psi(t) = 0$ for $t > 1 + \varepsilon$, $0 \leq \psi(t) \leq 1/\omega_d$ for $1 - \varepsilon \leq t \leq 1 + \varepsilon$, and $\int_{\mathbb{R}^d} \psi(|x|) dx = 1$. We have

$$\frac{1}{R^d} \int_{\mathbb{R}^d} \psi(|x|/R) e^{-2\pi i \langle x, \lambda \rangle} d\mu(x) = \{R^{-d} \psi(|x/R|) \mu(x)\}^\wedge(\lambda) = \hat{\psi}(R(\lambda - y)) \star \hat{\mu}(y),$$

then

$$(3) \quad \hat{\psi}(R(\lambda - y)) \star \hat{\mu}(y) = \int_{|\lambda - y| < 1} \hat{\psi}(R(\lambda - y)) d\hat{\mu}(y) + \int_{|\lambda - y| \geq 1} \hat{\psi}(R(\lambda - y)) d\hat{\mu}(y).$$

Note that $\hat{\psi}(0) = 1$ and $\hat{\psi}(R(\lambda - y)) \rightarrow 0$ for $\lambda \neq y$ as $R \rightarrow \infty$. By the Dominated Convergence Theorem, the first integral in (3) tends to 0 if $\hat{\mu}(\{\lambda\}) = 0$ and to $\hat{\mu}(\{\lambda\})$ otherwise. Since $\hat{\psi} \in S(\mathbb{R}^d)$, we see that the second integral in (3) does not exceed

$$c_N \int_{|\lambda-y| \geq 1} (R|y-\lambda|)^{-N} |\hat{\mu}|(y) = c_N R^{-N} \int_1^\infty t^{-N} d|\hat{\mu}|(B(\lambda, t)).$$

If N is large enough, then $|\hat{\mu}|(B(\lambda, t)) = O(t^{N-1})$ as $t \rightarrow \infty$. Arguing as in the previous lemma, we obtain that the second integral in (3) tends to 0 as $R \rightarrow \infty$.

Furthermore, since the measure μ is translation bounded, we get for large R

$$\left| \int_{\mathbb{R}^d} \psi(|x|/R) e^{-2\pi i \langle \lambda, x \rangle} d\mu(x) - \omega_d^{-1} \int_{B(x, R)} e^{-2\pi i \langle \lambda, x \rangle} d\mu(x) \right| \leq C\varepsilon R^d.$$

Since ε is arbitrary, we get the assertion of the Lemma. \blacksquare

Remark. We used property of μ to be translation bounded only in the latter part of the proof.

Proof of Theorem 11. Suppose that μ is translation bounded measure with countable spectrum and slowly increasing measure $\hat{\mu}$. By Lemma 1, $\psi \star \mu(s)$ is almost periodic for every $\psi \in S(\mathbb{R}^d)$. Check that $(\phi \star \mu)(t)$ is almost periodic for each continuous function ϕ with a compact support in the ball $B(R)$. Let $\psi_n \in S(\mathbb{R}^d)$, $\text{supp } \psi_n \subset B(R+1)$, be a sequence that uniformly converges to ϕ . Since μ is translation bounded, the almost periodic functions $(\psi_n \star \mu)(t)$ uniformly converge to $(\phi \star \mu)(t)$, therefore the latter function is almost periodic, and μ is an almost periodic measure.

Now suppose that μ is almost periodic. By Theorem 9, μ is translation bounded. Let ν be the atomic component of measure $\hat{\mu}$, let μ_1 be the inverse Fourier transform of ν , and μ_2 be the inverse Fourier transform of $\hat{\mu} - \nu$. It follows from the first part of the proof that μ_1 is an almost periodic measure. Therefore, μ_2 is almost periodic too. Let φ be C^∞ -function with compact support. The convolution $\varphi \star \mu_2$ is an almost periodic function, hence measure $\varphi \star \mu_2(x) m_d(x)$ is translation bounded. By Lemma 2, Fourier coefficient of $\varphi \star \mu_2$ at each frequency $\lambda \in \mathbb{R}^d$ equals $(\varphi \star \mu_2)^\wedge(\{\lambda\}) = \hat{\varphi}(\lambda) \hat{\mu}_2(\{\lambda\}) = 0$. Therefore, $\varphi \star \mu_2 \equiv 0$ for each $\varphi \in S(\mathbb{R}^d)$ with compact support. Then $\mu_2 = 0$ and $\hat{\mu} = \nu$. \blacksquare

To prove Theorem 2 we need two lemmas.

Lemma 3. Let $\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda$, $|a_\lambda| \geq \varepsilon > 0$, be an almost periodic measure on \mathbb{R}^d with uniformly discrete Λ . Then the set Λ is almost periodic too.

Proof. Check that for every continuous function $\varphi(x)$ with compact support and every $\varepsilon > 0$ the function $\sum_{\lambda \in \Lambda} \varphi(x - \lambda)$ has a relatively dense set of ε -almost periods. Set

$$\eta = (1/2) \inf_{\lambda, \lambda' \in \Lambda, \lambda \neq \lambda'} |\lambda - \lambda'|, \quad \beta = \inf_{\lambda \in \Lambda} |a_\lambda|.$$

Without loss of generality suppose that $\text{supp } \varphi \subset B(\eta)$, and $0 \leq \varphi(x) \leq \varphi(0) = 1$. There exists $\rho \in (0, \eta)$ such that the inequality $|x - x'| < \rho$ implies $|\varphi(x) - \varphi(x')| < \varepsilon$. Set $\psi(x) = \varphi(\eta x / \rho)$. Note that if $\psi(x) \neq 0$, then $|x| < \rho$. Let τ be β -almost period of the function $\psi \star \mu(x) = \sum_{\lambda \in \Lambda} a_\lambda \psi(x - \lambda)$. We have for all $x \in \mathbb{R}^d$

$$\left| \sum_{\lambda \in \Lambda} a_\lambda \psi(x - \lambda) - \sum_{\lambda' \in \Lambda} a_{\lambda'} \psi(x + \tau - \lambda') \right| < \beta.$$

Clearly, at most one term in every sum does not vanish. For $x = \lambda$ we get $|a_\lambda - a_{\lambda'} \psi(\lambda + \tau - \lambda')| < \beta$, hence $\psi(\lambda + \tau - \lambda') \neq 0$ and $|\lambda + \tau - \lambda'| < \rho$. Therefore, $|\varphi(x - \lambda) - \varphi(x + \tau - \lambda')| < \varepsilon$.

If there is another $\lambda'' \in \Lambda$ such that $\varphi(x + \tau - \lambda'') \neq 0$, then $|x + \tau - \lambda''| < \eta$ and $|\lambda' - \lambda''| < 2\eta$. This is impossible, therefore for all $x \in \mathbb{R}^d$

$$\left| \sum_{\lambda \in \Lambda} \varphi(x - \lambda) - \sum_{\lambda' \in \Lambda} \varphi(x + \tau - \lambda') \right| < \varepsilon.$$

Since there is a relatively dense set of β -almost periods τ of the function $\psi \star \mu$, we conclude that the function $\sum_{\lambda \in \Lambda} \varphi(x - \lambda)$ is almost periodic too. \blacksquare

Lemma 4. *Let $\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda$ be a uniformly discrete measure from $S'(\mathbb{R}^d)$ with the slowly increasing measure $\hat{\mu}$. Then $\sup_{\lambda \in \Lambda} |a_\lambda| < \infty$ and, consequently, the measure μ is translating bounded.*

Proof. Set $\eta = (1/2) \inf_{\lambda, \lambda' \in \Lambda, \lambda \neq \lambda'} |\lambda - \lambda'|$. Let $\psi(|y|)$ be a C^∞ -function such that $\text{supp } \psi(|y|) \subset B(\eta)$ and $\psi(0) = 1$. We have

$$\sup_{\lambda \in \Lambda} |a_\lambda| \leq \sup_{x \in \mathbb{R}^d} \left| \int \psi(|x - \lambda|) d\mu(\lambda) \right| = \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} \hat{\psi}(y) e^{2\pi i \langle x, y \rangle} d\hat{\mu}(y) \right| \leq \int_{\mathbb{R}^d} |\hat{\psi}(y)| d|\hat{\mu}|(y).$$

Taking into account that $\hat{\psi}(y) \in S(R^d)$ and arguing as above in Lemma 1, we obtain that the latter integral is finite. \blacksquare

Our proof of Theorem 2 is based on the following result.

Theorem 12 ([3]). *Let Λ be an almost periodic set in \mathbb{R}^d with a discrete set of differences $\Lambda - \Lambda$. Then Λ is a finite union of translates of a single full-rank lattice.*

Proof of Theorem 2. Since the set $\Lambda - \Lambda$ is discrete, we get that Λ is uniformly discrete. By Lemma 4, μ is translation bounded. By Theorem 11, μ is almost periodic. Lemma 3 implies that Λ is almost periodic as well. Now, using Theorem 12, we obtain the assertion of Theorem 2. \blacksquare

There is an analog of Theorem 12 for pairs of measures.

Theorem 13. *Let μ_1, μ_2 be large Fourier quasicrystals such that the set $\Lambda_1 - \Lambda_2$ of differences of their supports Λ_1, Λ_2 is discrete. Then there exists a full-rank lattice L such that the both supports Λ_1, Λ_2 are finite unions of translates of L .*

In fact, the proof of this theorem is the same as in the previous one. But instead of Theorem 12 we need to use the corresponding assertion for pairs of measures from [4].

3. WIENER'S THEOREM FOR QUASICRYSTALS

In this section we prove Theorems 7 and 8. We begin with the following lemma.

Lemma 5. *Let $F(\theta, \tau)$, $\theta = (\theta_1, \dots, \theta_N) \in [0, 1]^N$, $\tau \in [0, 1]$, be C^∞ -differentiable function in all variables and periodic with period 1 in each θ_j , $j = 1, \dots, N$. Then its Fourier series*

$$F(\theta, \tau) = \sum_{n \in \mathbb{Z}^N} c_n(\tau) e^{2\pi i \langle \theta, n \rangle}$$

converges absolutely and uniformly in $\tau \in [0, 1]$.

Proof. We have

$$c_n(\tau) = \int_{[0, 1]^N} F(\theta, \tau) e^{-2\pi i \langle \theta, n \rangle} d\theta.$$

Let $n = (n_1, \dots, n_N)$ with $n_1 \neq 0$. Integrating by parts in variable θ_1 two times, we get

$$c_n(\tau) = \frac{-1}{4\pi^2 n_1^2} \int_{[0, 1]^N} \frac{\partial^2 F(\theta, \tau)}{\partial \theta_1^2} e^{-2\pi i \langle \theta, n \rangle} d\theta.$$

If $n_1 \cdot n_2 \cdots n_m \neq 0$, then after integrating by parts in variables $\theta_1, \dots, \theta_m$ we get

$$c_n(\tau) = \frac{(-1)^m}{(2\pi)^{2m} n_1^2 \cdots n_m^2} \int_{[0,1]^N} \frac{\partial^{2m} F(\theta, \tau)}{\partial \theta_1^2 \cdots \partial \theta_m^2} e^{-2\pi i \langle \theta, n \rangle} d\theta.$$

Therefore,

$$|c_n(\tau)| \leq \frac{1}{(2\pi)^{2m} n_1^2 \cdots n_m^2} \max_{\theta, \tau} \left| \frac{\partial^{2m} F(\theta, \tau)}{\partial \theta_1^2 \cdots \partial \theta_m^2} \right|.$$

Consequently, for all $n \in \mathbb{Z}^N$ we get

$$|c_n(\tau)| \leq C(F) \min\{1, n_1^{-2}\} \cdots \min\{1, n_N^{-2}\}.$$

Obvious convergence of the series $\sum_{n \in \mathbb{Z}^k} \min\{1, n_1^{-2}\} \cdots \min\{1, n_N^{-2}\}$ implies the assertion of the Lemma. \blacksquare

Proof of Theorem 7. Let U be a neighborhood of K such that the function $h(z)$ is holomorphic and bounded on U . Set $\eta = (1/4) \text{dist}(K, \mathbb{C} \setminus U)$. Let $\varphi(|z|)$ be C^∞ -differentiable nonnegative function with support in $B(\eta)$ such that $\int_{B(\eta)} \varphi(|z|) dm_2(z) = 1$. Then the function

$$H(z) = \int_{\mathbb{R}^d} h(z - \zeta) \varphi(|\zeta|) dm_2(\zeta)$$

is C^∞ -differentiable on \mathbb{R}^d and coincides with $h(z)$ on the set $\{z : \text{dist}(z, K) \leq 3\eta\}$.

Suppose $f(x) = \sum_n c_n e^{2\pi i \langle x, \gamma_n \rangle}$. Choose $N < \infty$ such that for $S(x) = \sum_{n \leq N} c_n e^{2\pi i \langle x, \gamma_n \rangle}$ we get $\|f(x) - S(x)\|_W < \eta$. Applying Lemma 5 to the function

$$F(\theta, \tau) = H \left(\sum_{n \leq N} c_n e^{2\pi i \theta_n} + 2\eta e^{2\pi i \tau} \right)$$

and replacing θ_j with $\langle x, \gamma_j \rangle$, $j = 1, \dots, N$, we get

$$(4) \quad H(S(x) + 2\eta e^{2\pi i \tau}) = \sum_n c_n(\tau) e^{2\pi i \langle x, \rho_n \rangle} \in W, \quad \rho_n \in \text{Lin}_{\mathbb{Z}}\{\lambda_1, \dots, \lambda_N\},$$

and $\|H(S(x) + 2\eta e^{2\pi i \tau})\|_W \leq C(N)$ uniformly in $\tau \in [0, 1]$.

Next, taking into account that $|f(x) - S(x)| \leq \|f - S\|_W < \eta$, we have

$$(5) \quad (2\eta e^{2\pi i \tau} - [f(x) - S(x)])^{-1} = \sum_{k=0}^{\infty} \frac{[f(x) - S(x)]^k}{[2\eta e^{2\pi i \tau}]^{k+1}}.$$

Clearly, this sum belongs to W and its norm is uniformly bounded in $\tau \in [0, 1]$. Hence the same assertion is valid for the product of (4) and (5). Consequently, the function

$$g(x) = \int_0^1 \frac{H(S(x) + 2\eta e^{2\pi i \tau}) 2\eta e^{2\pi i \tau}}{(2\eta e^{2\pi i \tau} - [f(x) - S(x)])} d\tau = \frac{1}{2\pi i} \int_{|\zeta - S(x)|=2\eta} \frac{H(\zeta) d\zeta}{\zeta - f(x)}.$$

belongs to W .

If $f(x) \in K$, then $\text{dist}(S(x), K) < \eta$ and $B(S(x), 2\eta) \subset \{\zeta : \text{dist}(\zeta, K) < 3\eta\}$. Moreover, $f(x) \in B(S(x), 2\eta)$ and the function $H(z)$ coincides with $h(z)$ for $z \in B(S(x), 2\eta)$. Therefore, $g(x) = h(f(x))$ in this case. \blacksquare

To deduce Theorem 8 from Theorem 7 we prove the following two lemmas.

Lemma 6. Let μ be a slowly increasing measure with a translation bounded measure $\hat{\mu}$. Then for all $f \in W$ the Fourier transform $(f\mu)^\wedge$ is a translation bounded measure.

Proof. It is evident that $f\mu$ is a slowly increasing measure. Set $\mu_\gamma(x) = e^{2\pi i \langle x, \gamma \rangle} \mu(x)$ for $\gamma \in \mathbb{R}^d$. Clearly, for each $y \in \mathbb{R}^d$ we get $|\hat{\mu}_\gamma|(B(y, 1)) = |\hat{\mu}|(B(y - \gamma, 1))$. Suppose $f(x) = \sum_n c_n e^{2\pi i \langle x, \gamma_n \rangle} \in W$. Then $(f\mu)(x) = \sum_n c_n \mu_{\gamma_n}(x)$ and $(f\mu)^\wedge = \sum_n c_n \hat{\mu}_{\gamma_n}(x)$. Therefore for each $y \in \mathbb{R}^d$

$$|(f\mu)^\wedge|(B(y, 1)) \leq \sum_n |c_n| |\hat{\mu}_{\gamma_n}|(B(y, 1)) \leq \sum_n |c_n| \sup_{y \in \mathbb{R}^d} |\hat{\mu}|(B(y, 1)). \quad \blacksquare$$

Lemma 7. Let $\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda$ be a uniformly discrete measure with countable spectrum and slowly increasing Fourier transform. Then there is a function $f \in W$ such that $f(\lambda) = a_\lambda$ for all $\lambda \in \Lambda$.

Proof. Set $\eta = (1/2) \inf_{\lambda, \lambda' \in \Lambda, \lambda \neq \lambda'} |\lambda - \lambda'|$. Let ψ be C^∞ -differentiable function with support in $B(\eta)$ such that $\psi(0) = 1$. Then the function $f(x) = \psi \star \mu(x) = \sum_{\lambda \in \Lambda} a_\lambda \psi(x - \lambda)$ satisfies the condition $f(\lambda) = a_\lambda$ for all $\lambda \in \Lambda$. By Lemma 1, $f \in W$. \blacksquare

Proof of Theorem 8. We apply Theorem 7 with $K = \{a_\lambda : \lambda \in \Lambda\}$ and $f \in W$ such that $f(\lambda) = a_\lambda$, which exists by Lemma 7, and then we apply Lemma 6 for the measure μ and the function g from Theorem 7.

4. GENERALIZATION OF MEYER'S THEOREM

To prove the results of this section we need some definitions.

Lattice is a discrete subgroup of \mathbb{R}^d . If A is a lattice or a coset of some lattice in \mathbb{R}^d , then $\dim A$ is the dimension of the smallest translated subspace of \mathbb{R}^d that contains A . Every lattice L of dimension k has the form $T\mathbb{Z}^k$, where $T : \mathbb{Z}^k \rightarrow \mathbb{Z}^d$ is a linear operator of rank k . For $k = d$ we get a full-rank lattice. Also, the coset ring of an abelian topological group G is the smallest collection of subsets of G , that is closed under finite unions, finite intersections and complements and which contains all cosets of all open subgroups of G . Next, Bohr compactification \mathfrak{R} of \mathbb{R}^d is a compact group, its dual is \mathbb{R}^d with the discrete topology, \mathbb{R}^d is a dense subset of \mathfrak{R} with respect to the topology on \mathfrak{R} , and restrictions to \mathbb{R}^d of continuous functions on \mathfrak{R} are just almost periodic functions on \mathbb{R}^d , in particular, they are bounded and continuous on \mathbb{R}^d (see for example [13]).

Both Meyer's Theorem 3 and our Theorem 5 are based on Cohen's Idempotent Theorem:

Theorem 14 ([1]). Let G be a locally compact abelian group and \hat{G} its dual group. If ν is a finite Borel measure on G and is such that its Fourier transform $\hat{\nu}(\lambda) \in \{0, 1\}$ for all $\lambda \in \hat{G}$, then the set $\{\lambda : \hat{\nu}(\lambda) = 1\}$ is in the coset ring of \hat{G} .

We will apply Theorem 14 to Bohr compactification \mathfrak{R} of \mathbb{R}^d and its dual \mathbb{R}^d with the discrete topology. Then we will use the following theorem by M.Kolountzakis:

Theorem 15 ([5]). Elements of the ring of cosets of \mathbb{R}^d in the discrete topology, which are discrete in the usual topology of \mathbb{R}^d , are precisely finite unions of sets of the type

$$(6) \quad A \setminus (\cup_{j=1}^N B_j), \quad A, B_j \text{ discrete cosets, } \dim B_j < \dim A \text{ for all } j.$$

Proof of Theorem 5. Let ρ be the inverse Fourier transform of the measure μ , let $\psi(|x|) \in S(\mathbb{R}^d)$ be the function with compact support from Lemma 2 with $\varepsilon = 1/2$.

Set $\rho_R(y) = R^{-d} \psi(|y/R|) \rho(y)$. Integrating by parts and using (1), we get

$$|\rho_R| \leq R^{-d} \int_0^{(3/2)R} \psi(t/R) d|\rho|(B(t)) \leq R^{-d} \left[C' + C'' R^{-1} \int_0^{(3/2)R} t^d |\psi'(t/R)| dt \right].$$

Therefore,

$$\limsup_{R \rightarrow \infty} |\rho_R| \leq C < \infty.$$

Arguing as above in Lemma 2, we get that $\hat{\rho}_R(x)$ tends to $\hat{\rho}(\{\lambda\}) = \mu(\{\lambda\}) = a_\lambda$ for $x = \lambda \in \Lambda$ and to zero for $x \notin \Lambda$.

Since variations of the measures ρ_R are uniformly bounded, they act on all bounded functions on \mathbb{R}^d , and hence also on all functions from $C(\mathfrak{R})$. Therefore there exists a measure \mathfrak{r} on \mathfrak{R} with a finite total variation $|\mathfrak{r}|$, and a subsequence R' such that $\rho_{R'} \rightarrow \mathfrak{r}$ in the weak-star topology. In other words, $\langle \rho_{R'}, f \rangle \rightarrow \langle \mathfrak{r}, f \rangle$ as $R' \rightarrow \infty$ for all $f \in C(\mathfrak{R})$. Applying this to every character of \mathfrak{R} in place of f we obtain

$$\hat{\mathfrak{r}}(x) = \lim_{R' \rightarrow \infty} \hat{\rho}_{R'}(x) = \begin{cases} a_\lambda, & x = \lambda \in \Lambda, \\ 0, & x \notin \Lambda. \end{cases}$$

Note that $\hat{\mathfrak{r}}(x)$ is a continuous function with respect to the discrete topology on \mathbb{R}^d , and $|a_\lambda| \leq |\mathfrak{r}|$ for all a_λ .

Using Lemma 7, choose a function $f \in W$ such that $f(\lambda) = a_\lambda$ for all $\lambda \in \Lambda$. Taking into account that $\inf_\lambda |a_\lambda| > 0$ and using Theorem 7 with $h(z) = 1/z$, we construct the function $g(x) = \sum_n c_n e^{2\pi i \langle x, \tau_n \rangle} \in W$ with the property $g(\lambda) = 1/a_\lambda$. Since $\hat{\mathfrak{r}}(y + \tau) = e^{2\pi i \langle y, \tau \rangle} \hat{\mathfrak{r}}$, we see that the Fourier transform of a finite sum $\sum_{n \leq N} c_n \mathfrak{r}(y + \tau_n)$ is equal to $\sum_{n \leq N} c_n e^{2\pi i \langle y, \tau_n \rangle} \hat{\mathfrak{r}}$. The variation of the measure $\sum_{n \leq N} c_n \mathfrak{r}(y + \tau_n)$ does not exceed $\sum_{n \leq N} |c_n| |\mathfrak{r}|(\mathfrak{R})$, therefore the previous assertion is valid for infinite sums. Hence the Fourier transform of the measure $\mathfrak{b} = \sum_n c_n \mathfrak{r}(y + \tau_n)$ equals 1 for $x = \lambda \in \Lambda$ and 0 for $x \notin \Lambda$. By Theorem 14, Λ is in the coset ring of \mathbb{R}^d in discrete topology. Taking into account Lemma 4 and Theorem 11, we see that the measure μ is almost periodic. Therefore, by Lemma 3, measure $\sum_{\lambda \in \Lambda} \delta_\lambda$ is also almost periodic.

Using (6) and the notion of almost periodicity, it is easy to get the assertion of our theorem. By definition, put $\delta_A = \sum_{x \in A} \delta_x$ for a countable set $A \subset \mathbb{R}^d$. We have

$$\Lambda = \cup_{k=1}^K A_k \cup (\cup_{j=1}^J B_j) \setminus \cup_{i=1}^I C_i,$$

where A_k, B_j, C_i are cosets such that $\dim A_k = d$, $\dim B_j < d$, $\dim C_i < d$. Let $A_1 = x_1 + L_1$, $A_2 = x_2 + L_2$ be two cosets of dimension d . If $\dim A_1 \cap A_2 = d$, then $A_1 \cap (\mathbb{R}^d \setminus A_2)$ is a finite union of disjoint cosets of $L_1 \cap L_2$, and the same is valid for $A_2 \cap (\mathbb{R}^d \setminus A_1)$. Therefore we can replace $A_1 \cup A_2$ by a finite sum of disjoint cosets A'_l , $\dim A'_l = d$, hence $\delta_{A_1 \cup A_2} = \sum_l \delta_{A'_l}$. For every two cosets H_1, H_2 with $H_1 \cap H_2 \neq \emptyset$ and $\dim(H_1 \cap H_2) < d$ we get $\delta_{H_1 \cup H_2} = \delta_{H_1} + \delta_{H_2} - \delta_{H_1 \cap H_2}$. Repeating this transformation for each pair of cosets with nonempty intersection, we obtain a representation of δ_Λ after a finite number of steps of the form

$$\delta_\Lambda = \sum_{s=1}^{N_1} \delta_{D_s} + \sum_{s=1}^{N_2} \delta_{E_s} - \sum_{s=1}^{N_3} \delta_{F_s},$$

where D_s, E_s, F_s cosets, $\dim D_s = d$, $\dim E_s < d$, and $\dim F_s < d$. Obviously, every coset D_s has d linearly independent periods, hence δ_{D_s} is an almost periodic measure, and so is $\sum_{s=1}^{N_1} \delta_{D_s}$. Therefore the measure

$$(7) \quad \sum_{s=1}^{N_2} \delta_{E_s} - \sum_{s=1}^{N_3} \delta_{F_s} = \delta_\Lambda - \sum_{s=1}^{N_1} \delta_{D_s}$$

is almost periodic too. But its support is contained in a finite union of hyperplanes and isn't relatively dense. Therefore the measure (7) is identically zero, and $\delta_\Lambda = \sum_{s=1}^{N_1} \delta_{D_s}$. Since the measure δ_Λ has the unit masses at every point of Λ , we see that the cosets D_s are disjoint and $\Lambda = \cup_{s=1}^{N_1} D_s$. ■

Proof of the Corollary. By Lemma 4, A is a bounded set. Apply Theorem 8 with a function $h(z)$ that equals 1 on a neighborhood U of A and 0 on $\mathbb{C} \setminus U$, and then Theorem 5 for $\sum_{\lambda \in \Lambda, a_\lambda \in A} a_\lambda \delta_\lambda$. ■

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